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Teaching mathematics through history: Some trigonometric concepts

(1) History can be useful to teach science

The History of Science can provide tools so that students will be able to achieve scientific concepts successfully. As John Heilbron said in his conference *History as a collaborator of Science* (2003):

> Historical material should have a prominent place in the pedagogy of Science; but not to recall the past for itself, or for anecdotes or sugar coating, but because, for some purposes, History may be the best way to teach Science.[1]

In fact, the History of Science is a very useful tool to help the comprehension of mathematical ideas and concepts. Also, the History of Science is a very useful tool to help the understanding of Mathematics as a useful, dynamic, humane, interdisciplinary and heuristic science.[2]

On the one hand, the History of Science can be used as an implicit resource in the design of a syllabus, to choose context, mathematical problems and auxiliary sources. In addition, History can be used to program the sequence of the learning of a mathematical concept or idea. However, students do not have to accurately follow the historical sequence of the evolution of an idea, it should be remembered that the historical process of the constitution of knowledge was collective and depended on several social factors. In the past, most of the mathematicians had the aim of solving specific mathematical problems and they spent a lot of years working on them. In contrast, students are learning these concepts for the first time and sometimes they do not have enough motivation. Nevertheless, the knowledge of the historical evolution of a mathematical concept will provide teachers with tools to help the students to understand these concepts. At the same time it may also indicate the way to teach them.[3]

On the other hand, the History of Science can also be used in an explicit way in the classroom analyzing historical texts.[4] History can be useful to make explicit the differences between two contexts or to introduce a concept from analyzing a historical text. Moreover the proof of the solution of any historical problem can help and motivate the study of other kinds of problems. In another way, History can improve students' integral formation through the celebration of conferences, study programmes,

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[1] This is a fragment of the conference pronounced by the historian of science John Heilbron on November 2003 in the Centre of Study of History of Sciences (CEHIC) at the Universitat Autònoma of Barcelona.

[2] See M. R. Massa, "Aportacions de la història de la matemàtica a l'ensenyament de la matemàtica", *Biaix* (2003), pp. 4–9.

[3] Similar ideas had already been proclaimed by outstanding mathematicians like H. Poincaré, F. Klein, J. Rey Pastor, P. Puig Adam …

[4] For example, in Catalonia, teachers can design optional credits on History of Science to complement the students' formation. Among several useful studies, see E. Hairer and G. Wanner, *Analysis by its history* (1996); H. N. Jahnke and others, *History of Mathematics and Education: Ideas and Experiences* (1996); O. Toeplitz, *The Calculus. A Genetic Approach*. (1963); C. Weeks, *History of Mathematics, histories of problems* (1997).

etc.[5] Finally, History can also be used to do research works that concretely, in Catalonia, students have to do at the end of the secondary school (18 years).[6]

We have been working for three years now on the project: "The beginning and the development of Trigonometry throughout different civilizations." The goal of this paper is to show a brief summary of the analyzed texts and to explain some experiences implemented in a classroom of a secondary school to illustrate how teachers can introduce some trigonometric ideas using the History of Science.

(2) Using history to teach trigonometry

In Spain every autonomous community is in charge of their own secondary and graduate education, so we only speak about Education in Catalonia. The Department of Education of the Government of Catalonia would like to implement some contents on the History of Science in the curriculum of secondary education. In this way, we are creating materials to be used in the classroom. Precisely our group[7] of History of Mathematics of the Barcelona Association for Teaching and Learning of Mathematics (ABEAM) is working on the project "The beginning and the development of Trigonometry throughout different civilizations", which analyses the history of Trigonometry and its implications for the classroom.[8] The group is subsidized by the *Institute of Science of Education* (ICE) of the University of Barcelona.

The aims of this project are: firstly, to provide teachers with a brief history of the beginning of Trigonometry, focusing on the trigonometric ideas taught in secondary school; secondly, to give extensive bibliography about the history of Trigonometry and its teaching methods; thirdly, to provide the education community with translations of outstanding trigonometric texts and finally, to draw up activities for students to show them different ways of reasoning. This research covers from Ancient times to the Regiomontanus period (1436–1476). In fact, there is a text by Zeller (1944) which covers from Regiomontanus to Bartholomeo Pitiscus (1561–1613).[9]

We are working through different threads: notation, evolution of the concept of the angle, tables of sinus, the sinus and cosines theorem… So in our analysis of texts we have studied the tables calculating methods, where and how trigonometric notation began, when the first definitions and theorems appear, what the relation between Trigonometry and Geometry was, or the relation between Trigonometry and Astronomy, when Trigonometry became an independent Science and also how Trigonometry was applied to other topics such as Navigation, Optics, and Astronomy. We also focus our attention on the mathematical reasoning of demonstrations in order to analyze historical texts in the classroom successfully.

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[5] For instance, the symposium of study of René Descartes' work that took place in Barcelona (1996) provided students with additional formation from the mathematical, philosophical, physic or historical point of view. In the European seventeenth-century subjected to exhausting religious conflicts and with the intuition that a new time had begun, human knowledge was opening to new exploratory possibilities. Philosophy, mathematics, physics, music and language will become the faces of a new geometrical figure of the modern rationality. Students discovered from an interdisciplinary perspective how biography, institutions and historical events in Descartes shape an intellectual atmosphere generating questions and reflexions from the same worries and problems. See M. R. Massa, J. Comas and J. Granados, *Ciència, filosofia i societat en René Descartes* (1996).

[6] The list of titles of these research works can be very long, for instance: Pythagoras and the music, On Fermat's theorem, On Pascal's Arithmetic Triangle, On the beginning of algebraic language, Women and science, On incommensurability problem, Scientific Revolutions,…

[7] The coordinator of the group is**:** Mª Rosa Massa Esteve and the other members of group are**:** Mª Àngels Casals Puit (IES Joan Corominas), Iolanda Guevara Casanova (IES Badalona VII), Paco Moreno Rigall (IES XXV Olimpíada) and Fàtima Romero Vallhonesta (Inspecció d'Educació).

[8] For additional ideas on trigonometry history see E. Maor, *Trigonometric delights* (1998); M. V. Villuendas, *La trigonometria europea en el siglo XI. Estudio de la obra de Ibn Muad* "*El Kitab mayhulat*" (1979), Tomo XIX; S. M. C. Zeller, *The Development of trigonometry from Regiomontanus to Pitiscus* (1944).

[9] On Pitiscus work, Zeller said in his book: "For clarity of ideas and simplicity of form the *Trigonometria* of Pitiscus in the editions of 1600–1612, is the most outstanding treatise of trigonometry developed before the introduction of logarithms" [Zeller, 1944, 112].

(3) A survey of the development of trigonometry

As Villuendas said (1979), no historian would dare to date the beginning of Trigonometric Science. Trigonometry came about, surely, through several ways and also associated to the development of other topics like Astronomy, Arithmetic, Geometry and later Algebra.

From the chronological point of view we can begin with Babylonian Mathematics (ca. 1.500 BC).[10] Babylonians wrote mathematics on argyle tables in cuneiform language. It seems that they had no idea about how to quantify the angles and so, nothing similar to Trigonometry appeared. We can only find a described list of Pythagorean triples[11] to solve right triangles. According to Hoyrup (2002) and Caveing (1994) there was not anything similar to the chords tables. In contrast, Zeller (1944) and Villuendas (1979) pointed out that Babylonians could have calculated chords tables although these have not been found. In fact, they could have known the techniques to construct the tables as can be deduced from their astronomic calculations.

Something different could be pointed out about Egyptian Mathematics (ca. 1.600 BC). Egyptian Mathematics was written on papyrus in hieratic style. The solved problems dealt with basic Arithmetic, calculating surfaces and volumes and construction techniques. For instance we can quote Rhind Papyrus (1.650 BC), Moscow Papyrus (1.850 BC), Reisner Papyrus (1.880 BC), Berlin Papyrus (1.850 BC), etc. In Rhind Papyrus the scribe explained that there was a copy of other papyrus written 200 years previously and where the mathematical knowledge till that moment was collected. Rhind Papyrus included 87 mathematic problems on Arithmetic, Geometry and Algebra. We can point out the problem number 56 where the ratio between the height of the pyramid and the half of the side of the base is calculated. The scribe called this value the *seqt* of the pyramid; we call it now the cotangent of the inclination angle, determined by the face and the base of the pyramid. This calculation was repeated in several problems and in different ways. This is the only relation we have found with Trigonometry.

In the Hellenistic and the Alexandrian period, Astronomy and Trigonometry are linked inextricably. From that time we have analyzed the text by Aristarchus of Samos (310–230 BC), *On Sizes and Distances of the Sun and Moon*.[12] It is an astronomical treatise which deals with geometric problems considering that some relations between angles and sides of a triangle, which now we call trigonometric ratios, were known. It is especially interesting the use of geometric theorems to approximate the sinus of small angles. This work was part of a collection of texts called *Little Astronomy* together with *Optic* by Euclid, *Sphaericae* by Theodosia and others. In this work, Aristarchus through 18 propositions showed three theses from six hypotheses on sizes and distances of the asters.

We have to mention *The Elements* of Euclid (ca. 300 BC). It is not a trigonometric text but it contains some geometrical propositions, like Pythagoras theorem, that were fundamental to construct the tables of chords in the beginnings of systematic trigonometry.

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[10] *See*, e. g., M. Caveing, *Essai sur le savoir mathématique dans la Mésopotamie et l'Égypte anciennes (La constitution du type mathématique de l'idéalité dans la pensée grecque, 1)* (1994); S. Couchoud, *Mathématiques Egyptiennes. Recherches sur les connaissances mathématiques de l'Egypte pharaonique* (1993); R. J. Gillings, *Mathematics in the time of the pharaohs* (1972); J. Hoyrup, "Babylonian mathematics" a *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, vol I, Grattan-Guinness (ed.) (1994), pp. 21–29; J. Hoyrup, *Lenghts, Widths, Surfaces. A Portrait of Old Babylonian Algebra in its Kin* (2002); O. Neugebauer, *The exact sciences in Antiquity* (1969); C. S. Roero, "Egyptian mathematics" a *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, vol I, Grattan-Guinness (ed.) (1994), pp. 33–45; G. Sarton, *Introduction to the History of Science*, vol I, (1968); B. L. Van Der Waerden, *Science Awakening* (1962).

[11] Pythagorean triples are integers a, b, c such that $a^2 = b^2 + c^2$.

[12] *See*, e. g., L. Bulmer-Thomas, "Menelaus of Alexandria", *Dictionary of Scientific Biography* (1968), pp. 296–302; M. Clagett, "Archimedes", *Dictionary of Scientific Biography* (1968), pp. 213–231; Euclid, *The Elements*, 3 vol., T. L. Heath (ed.) (1956); T. Heath, *Aristarchus of Samos, the ancient Copernicus* (1981, first ed. 1913); Ptolemy, *Almagest* (1984); C. Ptolemy, *Las hipotesis de los planetas* (1987); W. H. Stahl, "Aristarchus", *Dictionary of Scientific Biography* (1968), pp. 246–250; A. Szabo-E. Maule, *Les débuts de l'astronomie, de la geografie et de la trigonometrie chez les grecs* (1986) ; G. J. Toomer, « Hipparchus », *Dictionary of Scientific Biography* (1968), pp. 207–223.

Trigonometry, in the modern sense of the word, began with Hipparchus of Nicea (190–120 BC) considered the greatest astronomer of antiquity. He set up an astronomical observatory on the island of Rhodes and needed some ratios to do his calculations, so they calculated a table of these ratios himself. He wrote twelve books on the computation of chords in a circle, but all are lost.

Spherical trigonometry came with the work of Menelaus of Alexandria, who introduced the concept of spherical triangle and proved some theorems about such triangles analogous to those Euclid had proved for plane triangles.

Another influent text in the development of trigonometry that we have studied, is *Almagest* (*Megale Syntaxis*) the greatest work of Claudius Ptolemy (90–168). It includes in the chapters 10 and 11 of the first book, a systematic calculation of the chord lengths for the central angles corresponding to the sides of some regular inscribed polygons, which was fundamental for the development of trigonometry.[13]

However the essential step in the development of trigonometry took place with Arabic culture.[14] The principal contributions of Arabic civilization were: the introduction of the six trigonometric ratios from the demonstrations of Menelaus theorem in spherical triangles; the deduction of sinus theorem and tangent theorem in plane and spherical triangles and the construction of new trigonometric tables using lineal and quadratic interpolation.

As an example we can mention the work *Kitab al-sakl al-qatta* (1260, *Treatise on complete quadrilateral*) by Nasir al-Din al-Tusi (1201–1274), that contains a systematic treatment on plane trigonometry and over all spherical trigonometry. It is one of the first works where the resolution of triangles was studied in an independent way, not as an auxiliary discipline for the astronomy.

Later, Johann Muller (1436–1476), better known as Regiomontanus, presented twofold purpose in his trigonometric works, to construct a uniform foundation and a systematic ordering of trigonometric knowledge, *De Triangulis omnimodis* (1464, first edition in 1533) and to make observations to improve tables for trigonometric calculations, *Tabulae directionum* (1467).

His work *De triangulis Omnimodis*[15] (*On Triangles* (1460), published 1533) contains a systematic description of methods to solve plane and spherical triangles. It includes five books, two of them on plane trigonometry and the other on spherical trigonometry. This work represented a started point and at the same time an endpoint into the history of trigonometry because from this book there were published several works which dealt more accurate on trigonometry to now.

(4) Ideas and experiences in classroom

In order to implement an activity we begin with a brief presentation of the personage and the epoch. Then we situate his work in the history of trigonometry, we analyze the aims of the author and finally students are pushed to follow the reasoning of this work to provide new mathematical ideas and perspectives. We will describe three classroom activities implemented in the last cycle of compulsory education (14–16 year old), briefly.

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[13] On Hindu trigonometry we have not already worked but this research is in course.

[14] *See*, e. g., E. Ausejo, “Sobre los conocimientos trigonométricos en los libros del saber de Astronomia de Alfonso X el Sabio” (1983), pp. 5–36; N.A.D. Al-Tusi, *Traité du quadrilatère* (1891); M.A. Català, “El desarrollo del álgebra y la trigonometría durante los siglos XIII al XV” (1981), pp. 63–80; M. T. Debarnot, “Trigonométrie”, in *Histoire des sciences arabes, Mathématiques et physique*, R. Rashed (ed.), vol. 2, (1997); J. Hamadanizadeh, “The trigonometric tables of Al-Kashi in his Zij-I Khaqani” (1980), pp. 38–45; J. Samsó, *Estudios sobre ABU NASR MANSUR B. `ALI B. `IRAQ,* (1969); J. Samsó, “Notas sobre la trigonometria esférica de Ibn Muad”, in *Islamic Astronomy and Medieval Spain* (1980), pp. 61–68; J. Vernet, *Historia de la ciencia àrabe* (1981); H. Wussing, “La consolidación de la trigonometría como rama científica independiente” in *Lecciones de Historia de las Matemáticas,* (1998), pp. 86–88; A. P. Youschkevitch, *Les Mathématiques Arabes (VIII–XV siècles)* (1976).

[15] *See*, e. g., N.G. Hairetdinova, “On the Oriental Sources of the Regiomontanus’ trigonometrical treatise” (1971), pp. 61–66; N.G. Hairetdinova, “On Spherical Trigonometry in the Medieval Near East and in Europe” (1986), pp. 136–146; V.J. Katz, “The calculus of the trigonometric functions” (1987), pp. 311–324; Regiomontanus, *De triangulis omnimodis* (1967); E. Rosen, “Copernicus”, *Dictionary of Scientific Biography* (1968), pp. 401–411; R.P. Ross, “Oronce Fine’s *De sinibus libri II:* The First printed trigonometric treatise of the French Renaissance” (1975), pp. 379–386.

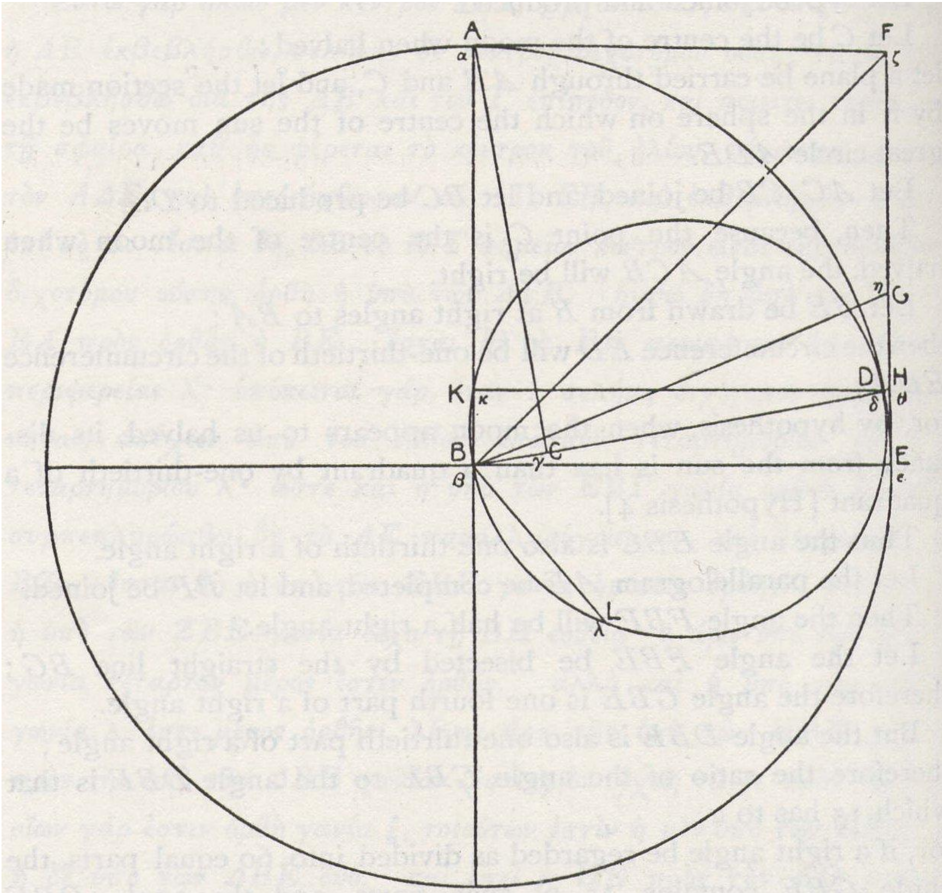
The first activity we are going to present, deals with the work *On Sizes and Distances of the Sun and Moon* by Aristarchus. In this activity, students are pushed to reproduce the proposition 7 which deal with the ratios between the distance of the Sun and the Moon from the Earth.

Aristarchus, Proposition 7

"The distance of the Sun from the Earth is greater than eighteen times, but less than twenty times, the distance of the moon from the Earth" (Heath, 1913, p. 377).

In fact, being B the centre of the Earth, A the centre of the Sun and C the centre of the Moon (figure 1), Aristarchus showed:

$$1/18 > \sin 3^{o} = CB : AB > 1/20.$$



He moved the problem from the triangle ABC to the constructed similar triangle BHE. He based his proof on the value of the angle ABC which he enounced as 87º and he had determined by observation .

The angle FBE is 45º and the angle GBE the half, then the ratio between two angles GBE and DBE, which is 3º, is GE : HE > (GBE) : (DBE) = 15 : 2

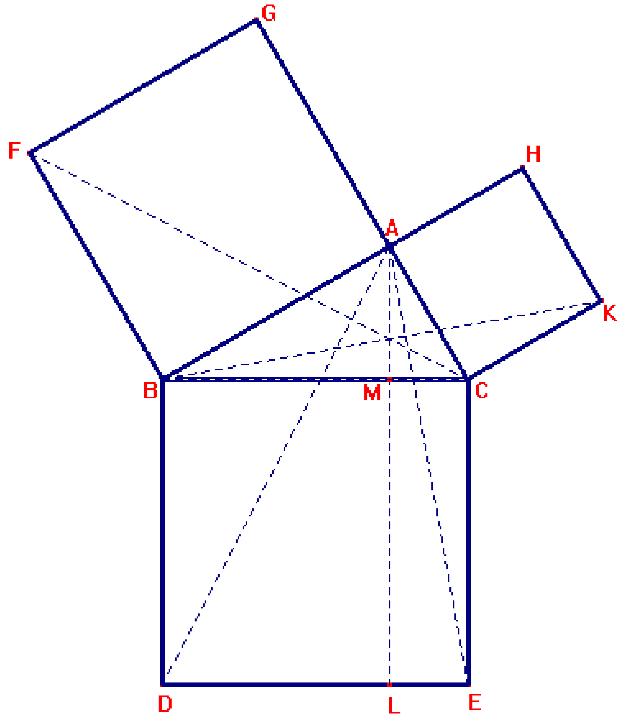
As $FB^2 = 2\ BE^2$,he used proportions in similar triangles and then $FG^2 = 2\ GE^2$. Using the inequality: 50 : 25 = 2 > 49 : 25. So FG : GE > 7 : 5 and FE : GE > 12 : 5 = 36 : 15.

So composing FE :GE with GE :HE obtain FE > 18 HE and BE>18 HE. Then, since the triangles ABC and BHE are similar, AB > 18 CB.

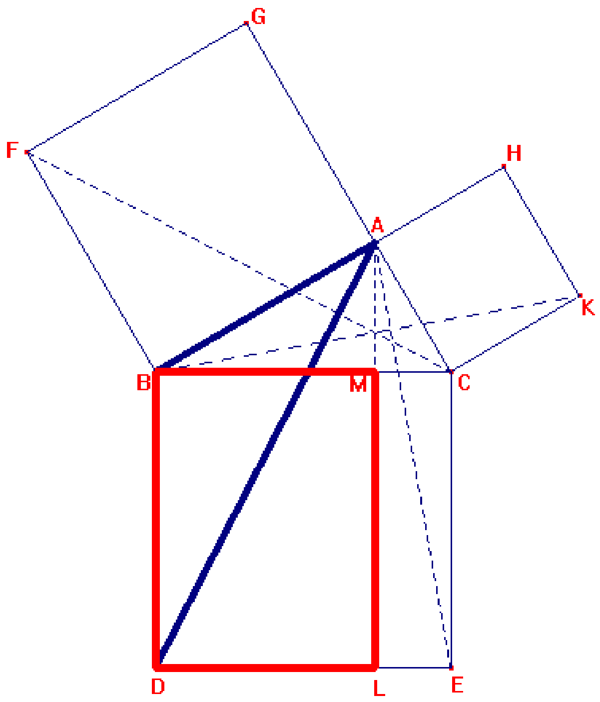
The second activity deals with propositions of Euclid's *Elements.* We have designed this classroom activity, using propositions I.47 and I.48 (Pythagoras theorem), and propositions II.12 and II.13 (cosines theorem). For instance we can read below a paragraph for Pythagoras Theorem.

Pythagoras Theorem

"In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle" (Heath, 1956, p. 349).



In the classroom activity, the students must fill one dossier with some steps that students have to follow. After every question we put an empty space that they have to fill. In some cases they also have to add some drawings. The conclusion should be something similar to the one written at the end of the next paragraph.

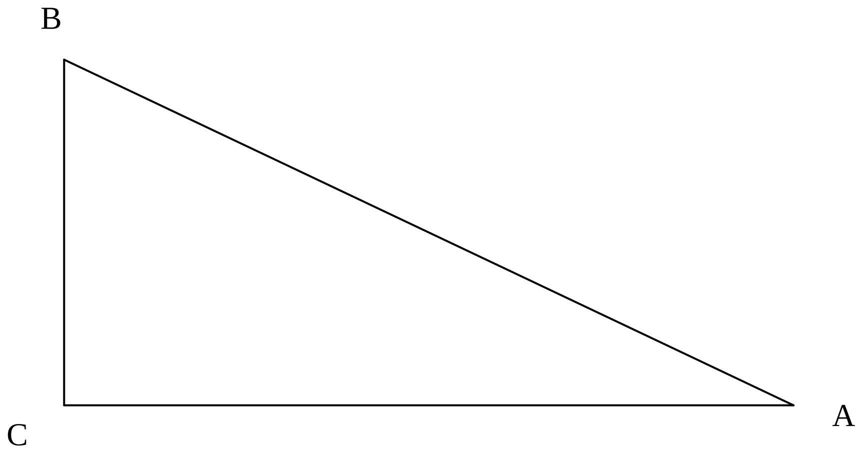


Students have to answer some questions as: If we take BD as a base of the triangle ABD, which line will be its height? And we can also take BD as a base of the parallelogram BL. In this case, which line will be its height? What is the relationship between the bases and heights of these two figures? And, consequently, what is the relationship between their areas? Students must conclude that the parallelogram is two times the triangle**.**

The third activity deals on the work *De triangulis omnimodis* (1464) by Regiomontanus. We have translated some theorems, from books I and II to design classroom activities.

Theorem 27 book I

"If two sides of a right triangle are given, all the angles can be found" (Regiomontanus (1967), pp. 64–67).



In the classroom activity, following Regiomontanus, students have to reproduce the demonstration in three cases: case 1, when BC and AB are given; case 2, when AC and AB are given; case 3, when BC and AC are given. Usually, given two sides of a right-angled triangle they have to find the angles using trigonometric ratios: sinus, cosinus and tangent, but in this Theorem Regiomontanus only uses the sinus and he moves the three cases to the first one with the sinus. We also ask students to compare this procedure using trigonometric tables with the procedure, used nowadays, with calculator.

Students have to answer some questions as: According to sinus definition, what does the quotient BC/AB mean? Regiomontanus looked at trigonometric tables, but you must work with calculator. Calculate BC/AB and search with the calculator the angle corresponding to this sinus. Looking at the figure, which angle has AC/AB as its sinus?

(5) Concluding remarks

From our experiences we can affirm that analyzing historical texts improve the students' integral formation giving them additional knowledge about social and scientific context of these periods and moreover designing activities related to several topics, geometry, trigonometry and algebra, may also contribute to improve the students mathematical formation.

According to these results we have created new teaching material which contains explanations that use a constructive learning method. It has been satisfactorily experimented with students who build their own reasoning like the old mathematics made.

So we can conclude that the History of Science can provide valuable tools to improve Mathematic training. However the difficulties to be implemented in the classroom are multiples. It is essential that Government took more seriously the formation in History of Science of professors, mathematicians, biologists, physicists…This situation will be improved with the program of course on the History of Science at the University…

Finally we would like to finish with a quotation of Pere Puig Adam, a great Catalan mathematician:

Science grows through mixed process of analysis and synthesis. All these process remain hidden in the synthetic expositions of Science used frequently in the teaching. Looking for an economic exposition which normally is more advantageous for who expose than for who receive it, it has been pointed more and more the separation between two processes that has never been separated: one, the process of the genesis of knowledge and the other the process of his transmission.