On Some Combinatorial Problems Concerning Partitions of a Box

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Abstract. In this paper we extend the results of [4]. We consider partitions of n-dimensional boxes in $\mathbf{R}^n$, $n \geq 2$ into a finite number of n-dimensional boxes with disjoint interiors. We study sets $X \subseteq (0,\infty)$ with the following Property (W_n): for every n-dimensional box P and every partition of P, if each constituent box has one side with the length belonging to X, then the length of one side of P belongs to X. We prove that if $\{G_\alpha\}_{\alpha \in \Gamma}$ is a chain of additive subgroups of $\mathbb{R}$, and each $I_\alpha \subseteq (0, \infty)$ $(\alpha \in \Gamma)$ is either $[a_\alpha, \infty)$ or (a_α, ∞), then the set $\bigcup_{\alpha \in \Gamma} G_\alpha \cap I_\alpha$ has Property (W_n). We prove that for every set $X \subseteq \{1, 2, 3, ...\}$ closed with respect to summation there exists a number $n(X) > 0$, such that for every $n \geq 2$, X fulfills a condition obtained from Property (W_n) by allowing only boxes P with all sides longer than $n(X)$.

In this paper we extend the results of [4] though our terminology differs from that of [4]. We consider partitions of n-dimensional boxes in $\mathbf{R}^n$, $n \geq 2$, into a finite number of n-dimensional boxes with disjoint interiors. The boxes constituting such a partition have sides parallel to the sides of the whole box. By a binary partition we mean a partition, which can be obtained by repeating the operation of partition of the box (whole in the first step and then its parts) to two smaller constituent boxes. We study sets $X \subseteq (0,\infty)$ with the property (W_n): for

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every n-dimensional box P and every partition of P, if each constituent box has one side with the length belonging to X, then the length of one side of P belongs to X.

Theorem 1. *If the set $X \subseteq (0, \infty)$ has Property (W_n), it also has the following properties:*

1) *X is an additive semigroup.*

2) *X is closed with respect to the operation $\delta(x, y, z) = x + y + z - 2 \cdot \min(x,y,z)$.*

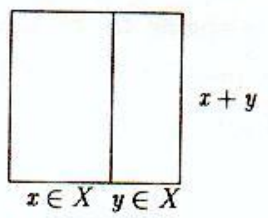
3) *For $c > 0$, the set cX has Property (W_n).*

4) *For $a > 0$, the sets $X \cup [a, \infty)$, $X \cup (a, \infty)$ have Property (W_n).*

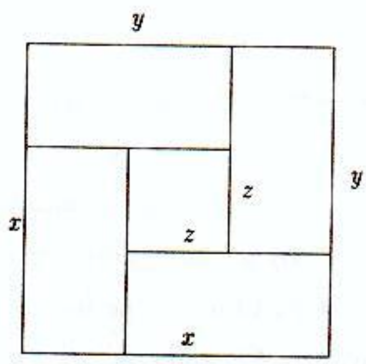
5) *For $a > 0$, the sets $X \cap [a, \infty)$, $X \cap (a, \infty)$ have Property (W_n).*

Proof. We begin with the case $n = 2$.

1) follows from the following bisection of a square



2) We can assume that $x > z$, $y > z$. The result then follows from the following partition of a square:



3) is obvious after performing on the rectangle P the similarity transformation with scale $\frac{1}{c}$.

4) We present the proof for a closed interval $[a, \infty)$. The proof when the interval is open (a, ∞) is analogous.

When every constituent rectangle has one side with its length belonging to X, then, by Property (W_2) for X, one side of the large rectangle belongs to $X \subseteq X \cup [a, \infty)$.

In the opposite case some constituent rectangle has one side not shorter than a, so one side of a large rectangle is not shorter than a.

5) We present (after Z. Moszner) the proof for a closed interval $[a, \infty)$. The argument for an open interval is analogous.

We assume that every constituent rectangle has one side belonging to $X \cap [a, \infty)$. Then one side of the large rectangle P is not shorter than a, and by Property (W_2) for X one side of the rectangle P belongs to X. When two sides of the rectangle P are not shorter than a, then one of them belongs to $X \cap [a, \infty)$. When only one side B of the rectangle P is not shorter than a, we consider rectangles P_k adjacent to B. Their sides perpendicular to B are shorter than a, hence their sides parallel to B belongs to $X \cap [a, \infty)$. Thus the length of B is a sum of lengths belonging to $X \cap [a, \infty)$. By 1), X is a semigroup, so we see that the length of B belongs to $X \cap [a, \infty)$.

As the proofs of items 1-4 for $n > 2$ are straightforward generalizations, we omit them here. We will prove item 5 by induction with respect to n. The case $n = 2$ is already proven. We present (extending Zenon Moszner'r proof for $n = 2$) the proof for a closed interval $[a,\infty)$. The argument for an open interval is analogous. The length of one side of a large n-dimensional box P belongs to X. Thus item 5 is true when all sides of P are not shorter than a. In the opposite case we choose a certain side shorter than a. We conside an $(n - 1)$-dimensional face of the box P orthogonal to this side and an induced partition of this face into $(n - 1)$-dimensional boxes. Each $(n - 1)$-dimensional box from the partition of this face has one side with its length belonging to $X \cap [a, \infty)$. By our induction assumption, the length of one side of this face belongs to $X \cap [a, \infty)$, which completes the proof. □

In [1] (see also [2]) N. G. de Bruijn proved a result about packing n-dimensional bricks into an n-dimensional box that, when $n = 2$, implies that if an $a \times b$ rectangle is covered with interior pairwise-disjoint copies of a $c \times d$ rectangle, then each of c, d divides one of a, b. De Bruijn's proof has been generalized to yield that the set of positive integers has property (W_2). This result has a variety of proofs (see [4]). Our Theorem 2 strengthens this result.

Theorem 2. *If $\{G_\alpha\}_{\alpha \in \Gamma}$ is a chain of additive subgroups of $\mathbb{R}$, and each $I_\alpha \subseteq (0, \infty)$ $(\alpha \in \Gamma)$ is either $[a_\alpha, \infty)$ or (a_α, ∞), then the set $\bigcup_{\alpha \in \Gamma} G_\alpha \cap I_\alpha$ has Property (W_n).*

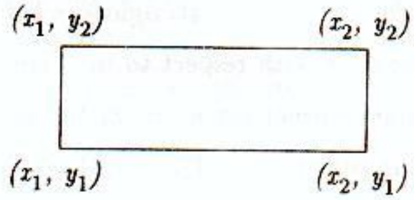
Proof. We begin with Lemma 1 (see also [4]) which follows.

Lemma 1. *The set $G \cap (0, \infty)$, where G is an additive subgroup of $\mathbb{R}$, has Property (W_2).*

Proof. We choose the cartesian coordinate system with the axis parallel to the sides of the rectangle. Let us denote by C the set of the coordinates of the vertices of the rectangles belonging to the partition. Since the relation $xRy \Leftrightarrow x - y \in G$ is an equivalence relation, there exists a function $f\colon C \to C$ which fulfills the condition: $\forall x \in C\ \forall y \in C\ (f(x) = f(y) \Leftrightarrow x - y \in G)$.

To the rectangle with vertices (x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), with $x_1 < x_2$, $y_1 < y_2$ we assign a number

$$
\begin{aligned}
m(P) &= (f(x_2) - f(x_1)) \cdot (f(y_2) - f(y_1)) \\
&= f(x_1) \cdot f(y_1) + f(x_2) \cdot f(y_2) - f(x_1) \cdot f(y_2) - f(x_2) \cdot f(y_1).
\end{aligned}
$$



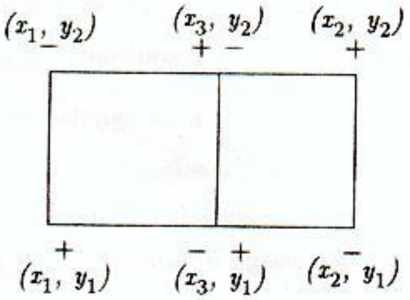
Hence, for each vertex with coordinates (x, y), the product $f(x) \cdot f(y)$ enters the sum with a "plus" sign when the diagonal is directed this way : /, and a "minus" sign for the diagonal directed the other way: \.

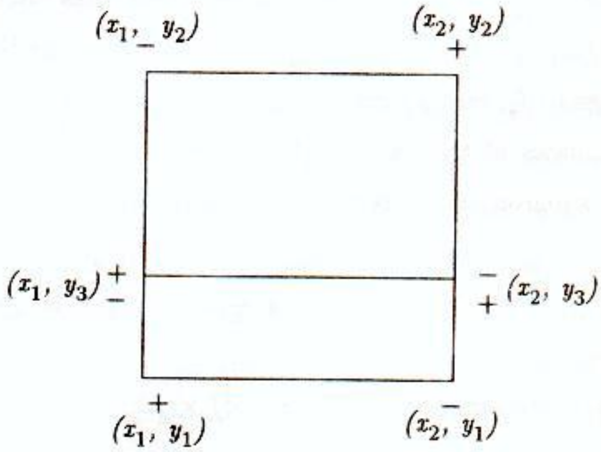
The thesis of Lemma 1 comes from Observations 1 and 2.

Observation 1. (Trivial). $m(P) = 0$ if and only if at least one side of the rectangle P has its length belonging to $G \cap (0,\infty)$.

Observation 2. *If* $P = P_1 \cup \ldots \cup P_n$ *is a partition of the rectangle* P, *then* $m(P) = \sum_{k=1}^{n} m(P_k)$.
Proof. 1) For a partition of a rectangle into two rectangles we prove our proposition by direct decomposition.

The figures show the possibilities (the signs cancel on the line of partition, we recall that the product $f(x) \cdot f(y)$ enters the sum with a "plus" sign when the diagonal is directed this way: /, and a "minus" sign for the diagonal directed the other way: \).





2) For binary partitions the proposition follows from 1).

3) When we have any non-binary partition, we can form the set C_x of the x-coordinates of the vertices of the rectangles entering the partition, and the set C_y of the y-coordinates of the vertices of the rectangles entering the partition.

The partition of the rectangle P formed by the straight lines $x = a$, $y = b$, where $a \in C_x$, $b \in C_y$, is a binary partition of the rectangle P, which was made by a partition of some P_k by binary partitions.

Hence replacing some $m(P_k)$ with the sums we obtain our proposition with the help of 2). This completes our proof of Lemma 1. □

Using similar methods we can prove the following additional theorem:

If G_1 and G_2 are additive subgroups of $\mathbb{R}$ and each constituent rectangle has either the height belonging to $G_1 \cap (0, \infty)$ or the width belonging to $G_2 \cap (0, \infty)$, then the whole rectangle has either the height belonging to $G_1 \cap (0, \infty)$ or the width belonging to $G_2 \cap (0, \infty)$. This generalization can be found also in [4].

Now we begin the actual proof of Theorem 2. We consider only finite partitions, hence we may consider only finite chains of groups of form $G_1 \subseteq G_2 \subseteq \ldots\ldots \subseteq G_m$. We can additionally assume that $I_1 \supset I_2 \supset \ldots\ldots \supset I_m$. We choose the coordinate system with the axes parallel to the sides of the large n-dimensional box P. For each axis x_i $(i = 1, \ldots\ldots, n)$ we denote by d_i the length of this side of P, which is parallel to the axis x_i. Let $S = \{i \in \{1, 2, \ldots, m\} : d_i \in I_1\}$. For $i \in S$ we choose the largest $v(i) \in \{1, 2, \ldots\ldots, m\}$ for which $d_i \in I_{v(i)}$. Every constituent box has one side with the length belonging to $(G_1 \cap I_1) \cup (G_2 \cap I_2) \cup \ldots\ldots \cup (G_m \cap I_m) \subseteq I_1$. Thus the length of a certain side of the large box P belongs to I_1, which implies $S \neq \emptyset$. For $i \in \{1, 2, \ldots\ldots, n\}$ we denote by C_i the set of the i-th coordinates of the vertices of the constituent boxes. For each $i \in \{1, 2, \ldots\ldots, n\}$ and an additive subgroup $G \subseteq \mathbf{R}$ there exists a function $f_G \colon C_i \to C_i$, which satisfies the condition:

$$\forall\, x \in C_i \quad \forall\, y \in C_i \quad (f_G(x) = f_G(y) \Leftrightarrow x - y \in G).$$

To an n-dimensional box $Q = [x_1^0, x_1^1] \times [x_2^0, x_2^1] \times \ldots\ldots \times [x_n^0, x_n^1]$ we assign a number

$$m(Q) = \prod_{i \in S} \left(f_{G_{v(i)}}(x_1^1) - f_{G_{v(i)}}(x_1^0) \right) \cdot \prod_{i \in \{1, \ldots, n\} \setminus S} (x_1^1 - x_1^0).$$

Repeating the argument from our Lemma 1 we can prove that this assignment is finitely additive.

We will prove that if $d \in (G_1 \cap I_1) \cup (G_2 \cap I_2) \cup \ldots\ldots \cup (G_m \cap I_m)$ is the length of the constituent box Q, which is parallel to x_i-axis, then $i \in S$, $d \in G_{v(i)} \cap (0, \infty)$. Indeed, since $d \le d_i$, we have $d_i \in I_1$. Thus $i \in S$. If $v(i) = m$, then $d \in (G_1 \cap I_1) \cup (G_2 \cap I_2) \cup \ldots\ldots \cup (G_m \cap I_m) = (G_1 \cap I_1) \cup (G_2 \cap I_2) \cup \ldots\ldots \cup (G_{v(i)} \cap I_{v(i)})$

$$\subseteq G_{v(i)} \cap (0, \infty).$$

If $v(i) < m$, then $d_i \notin I_{v(i)+1}$. Since $d \le d_i$ it follows that $d \notin I_{v(i)+1}$, $d \notin I_{v(i)+2}$, $\ldots\ldots$, $d \notin I_m$. Hence from the condition $d \in (G_1 \cap I_1) \cup (G_2 \cap I_2) \cup \ldots\ldots \cup (G_m \cap I_m)$, we see that $d \in (G_1 \cap I_1) \cup (G_2 \cap I_2) \cup \ldots\ldots \cup (G_{v(i)} \cap I_{v(i)}) \subseteq G_{v(i)} \cap (0, \infty)$. This means that $m(Q) = 0$ for all constituent boxes Q. By finite additivity of the assignment m we find that $m(P) = 0$. This implies that for some $i \in S$ $d_i \in G_{v(i)} \cap (0, \infty)$. Since $d_i \in I_{v(i)}$, we have $d_i \in G_{v(i)} \cap I_{v(i)} \subseteq (G_1 \cap I_1) \cup (G_2 \cap I_2) \cup \ldots\ldots \cup (G_m \cap I_m)$. This completes the proof. □

Let us denote by $\mathbb{Z}$ the additive group of integers. For the sets $X \subseteq \{1, 2, 3, \ldots\}$ we present partial sufficient condition for fulfilling Property (W_n). This result is contained in Theorem 3 which we prove by means of Lemmas 2 and 3.

Lemma 2 (see also [3]). *If d is the greatest common divisor of positive integers $a_1, a_2, \ldots\ldots, a_m$, then every sufficiently large positive integer divisible by d can be presented in the form $a_1x_1 + a_2x_2 + \ldots\ldots + a_mx_m$, where $x_1, x_2, \ldots\ldots, x_m$ are nonnegative integers.*

Proof. It is obvious that every positive integer divisible by d can be presented in the form $a_1y_1 + a_2y_2 + \ldots\ldots + a_my_m$, where $y_1, y_2, \ldots\ldots, y_m$ are integers. We put:

$$y_i = q_ia_m + x_i, \; 0 \leq x_i < a_m, \; q_i \in \mathbb{Z} \quad (i = 1, \ldots\ldots, m - 1).$$

From the following equality

$$\sum_{i=1}^{m-1} a_ix_i + a_m \cdot \left(y_m + \sum_{j=1}^{m-1} a_j\, q_j \right) = \sum_{i=1}^{m} a_i\, y_i$$

we obtain the result for positive integers divisible by d which are not less than $\sum_{i=1}^{m-1} a_ia_m$.

Lemma 3 (see also [3]). *For every set $X \subseteq \{1, 2, 3, \ldots\}$ closed with respect to summation there exists an $n(X) > 0$ and a subgroup $G \subseteq \mathbb{Z}$ for which $X \subseteq G \cap (0, \infty)$ and $X \cap (n(X), \infty) = G \cap (n(X), \infty)$.*

Proof. Let $X = \{a_1, a_2, a_3, \ldots\}$. We introduce the following notation: d denotes the greatest common divisor of numbers belonging to X, d_i denotes the greatest common divisor of numbers $a_1, a_2, \ldots\ldots, a_i$ and G denotes an additive group of integers divisible by d.

Of course $X \subseteq G \cap (0, \infty)$.

We have $d_1 \geq d_2 \geq d_3 \geq \ldots\ldots \geq d$. Then the sequence $\{d_i\}$ beginning from some term d_m is constant. Then $d_m = d$, hence d is the greatest common divisor of the numbers

$a_1, a_2, \ldots\ldots, a_m$.

By means of Lemma 2 every sufficiently large positive integer divisible by d can be presented as $a_1x_1 + a_2x_2 + \ldots\ldots + a_mx_m$, where $x_1, x_2, \ldots\ldots, x_m$ are nonnegative integers.

As the set X is closed with respect to summation, every $a_ix_i \neq 0$ can be obtained by multiple summing of a_i. Thus every sufficiently large positive integer divisible by d belongs to X. We have just proved that there exists a constant $n(X) > 0$ for which $X \cap (n(X), \infty) = G \cap (n(X), \infty)$. This completes the proof of Lemma 3. □

Now we can revert to Theorem 3.

Theorem 3. *For every set $X \subseteq \{1, 2, 3, \ldots\}$ closed with respect to summation there exists a number $n(X) > 0$, such that for every $n \geq 2$ X fulfills the condition obtained from Property (W_n) by restricting the boxes mentioned in Property (W_n) to those that have both sides longer than $n(X)$.*

Proof. Let the group G fulfill the assertion of Lemma 3. When the box P has all sides longer than $n(X)$ and every box from the partition of P has one side with length belonging to $X \subseteq G \cap (0, \infty)$, then by Theorem 2 the length of one side of the box P belongs to $G \cap (0, \infty)$. Since $G \cap (n(X), \infty) = X \cap (n(X), \infty)$, the length of this side belongs to X. □

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