A currently true statement Γ of the form "$\exists f:\mathbb{N}\to\mathbb{N}$ of unknown computability $\ldots$", where f is conjecturally uncomputable, Γ remains unproven when f is computable, and Γ holds forever when f is uncomputable

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Abstract

We define a function $\gamma:\mathbb{N}\to\mathbb{N}$ which eventually dominates every computable function $\alpha:\mathbb{N}\to\mathbb{N}$. We show that there is a computer program which for $n\in\mathbb{N}$ prints the sequence $\{\gamma_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $\gamma(n)$. We define a function $\beta:\mathbb{N}\to\mathbb{N}$ of unknown computability which eventually dominates every function $\delta:\mathbb{N}\to\mathbb{N}$ with a single-fold Diophantine representation. We show that there is a computer program which for $n\in\mathbb{N}$ prints the sequence $\{\beta_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $\beta(n)$. Let Γ denote the following statement: $\exists f:\mathbb{N}\to\mathbb{N}$ *of unknown computability such that* f *eventually dominates every function* $\delta:\mathbb{N}\to\mathbb{N}$ *with a single-fold Diophantine representation and there is a computer program which for* $n\in\mathbb{N}$ *prints the sequence* $\{f_i(n)\}_{i=0}^{\infty}$ *of non-negative integers converging to* $f(n)$. We show that Γ has all properties from the title of the article.

Key words and phrases: computable function, eventual domination, finite-fold Diophantine representation, limit-computable function, predicate $\mathcal{K}$ of the current mathematical knowledge, single-fold Diophantine representation, time-dependent truth in mathematics with the predicate $\mathcal{K}$ of the current mathematical knowledge.

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1 Predicate $\mathcal{K}$ of the current mathematical knowledge

In sections 1–4, $\mathcal{K}$ denotes both the predicate satisfied by every currently known theorem and the set of all currently known theorems. The set $\mathcal{K}$ is time-dependent. Observations 1–5 explain the definition of $\mathcal{K}$. This definition is intentionally simple, which restricts the set $\mathcal{K}$. Observations 1–4 hold at any time.

Observation 1. *Every written down theorem from past or present belongs to* $\mathcal{K}$*.*

Observation 2. *If* $\Delta\in\mathcal{K}$*, then every particular case of* Δ *belongs to* $\mathcal{K}$*. For example,*

$$\{\emptyset\cup\{n\}=\{n\}:\ n\in\mathbb{N}\}\subseteq\mathcal{K}$$

Observation 3. *Since the Replacement Axiom is a scheme of axioms, only finitely many axioms of* ZFC *belong to* $\mathcal{K}$*.*

Observation 4. *$\mathcal{K}$ is not deductively closed.*

Proof. There exists $n \in \mathbb{N}$ such that $\neg\mathcal{K}(\underbrace{(2 \cdot 2 = 4) \wedge \ldots \wedge (2 \cdot 2 = 4)}_{\text{conjunction of } n+3 \text{ statements}})$. □

Observation 5. *Currently,*

$$(\neg\mathcal{K}((((((((9!)!)!)!)!)!)!)!-\text{th digit of } \pi \text{ is even}))\wedge$$

$$(\neg\mathcal{K}((((((((9!)!)!)!)!)!)!)!-\text{th digit of } \pi \text{ is odd}))$$

Hence, $\mathcal{K}$ is not deductively closed.

2 A limit-computable function $\gamma : \mathbb{N} \to \mathbb{N}$ which is constructively defined and eventually dominates every computable function $\alpha : \mathbb{N} \to \mathbb{N}$

. For $n \in \mathbb{N}$, let

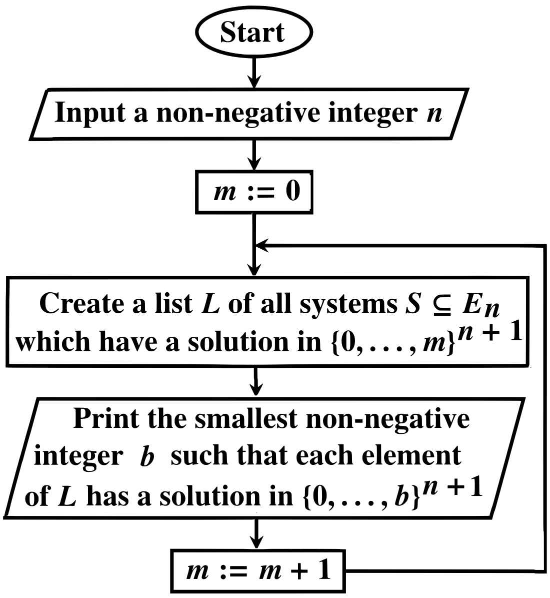
$$E_n = \{1 = x_k,\ x_i + x_j = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$$

Theorem 1. *([5, p. 118]). There exists a limit-computable function $\gamma : \mathbb{N} \to \mathbb{N}$ which eventually dominates every computable function $\alpha : \mathbb{N} \to \mathbb{N}$.*

We present an alternative proof of Theorem 1. For $n \in \mathbb{N}$, $\gamma(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then $\mathcal{S}$ has a solution in $\{0, \ldots, b\}^{n+1}$. The function $\gamma : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $\alpha : \mathbb{N} \to \mathbb{N}$, see [6].

Corollary 1. *For every $k \in \mathbb{N}$, the function $\mathbb{N} \ni n \to k + \gamma(n) \in \mathbb{N}$ is uncomputable.*

Flowchart 1 shows a semi-algorithm which computes $\gamma(n)$ in the limit, see [6].



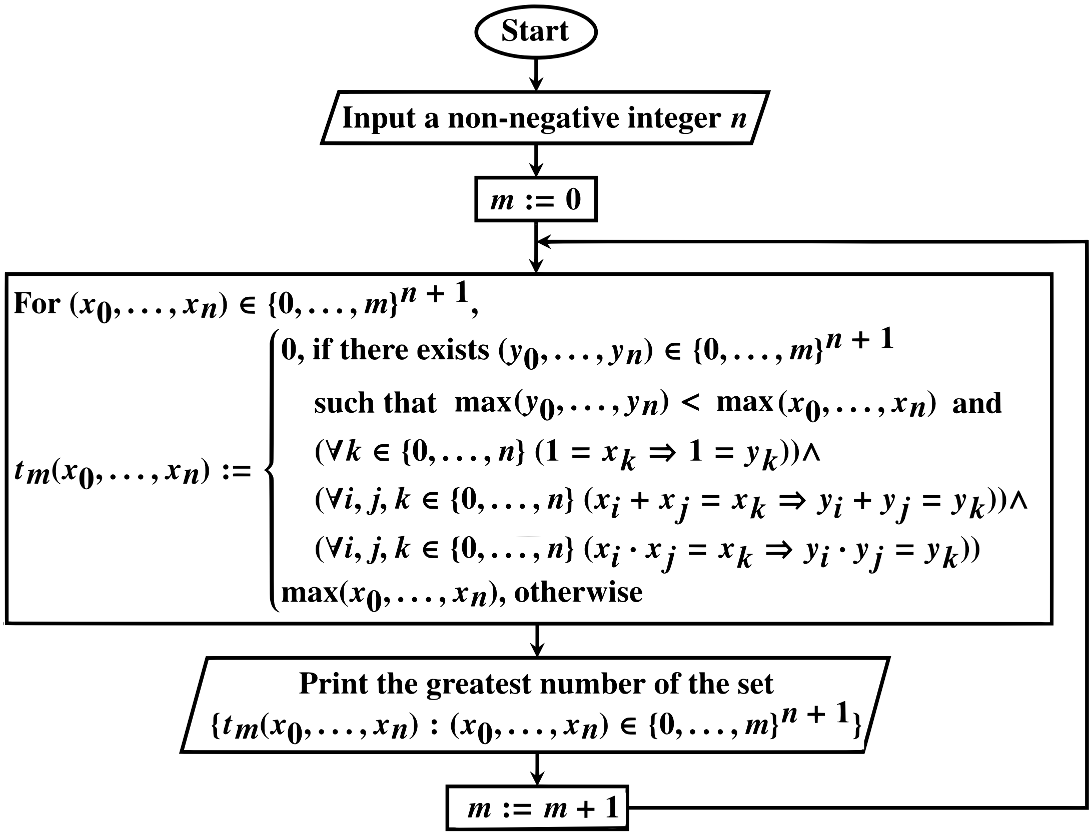
Flowchart 1

A semi-algorithm which computes $\gamma(n)$ in the limit

Proposition 1. *If $k \in \mathbb{N}$, then the statement "the function $\mathbb{N} \ni n \to k + \gamma(n) \in \mathbb{N}$ is uncomputable" belongs to $\mathcal{K}$.*

Proof. It follows from Corollary 1 and Observations 1 and 2. □

Flowchart 2 shows a simpler semi-algorithm which computes $\gamma(n)$ in the limit.



Flowchart 2

A simpler semi-algorithm which computes $\gamma(n)$ in the limit

Lemma 1. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 2 does not exceed the number printed by Flowchart 1.*

Proof. For every $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k :\ (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Lemma 2. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 1 does not exceed the number printed by Flowchart 2.*

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ and $(a_0, \dots, a_n)$ solves $\mathcal{S}$, then $(a_0, \dots, a_n)$ solves the system of equations

$$\widetilde{\mathcal{S}} := \{1 = x_k :\ (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Theorem 2. *For every $n, m \in \mathbb{N}$, Flowcharts 1 and 2 print the same number.*

Proof. It follows from Lemmas 1 and 2. □

3 A limit-computable function $\beta : \mathbb{N} \to \mathbb{N}$ of unknown computability which is constructively defined and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation

. The Davis-Putnam-Robinson-Matiyasevich theorem states that every listable set $\mathcal{M} \subseteq \mathbb{N}^n$ $(n \in \mathbb{N} \setminus \{0\})$ has a Diophantine representation, that is

$$(a_1, \ldots, a_n) \in \mathcal{M} \Longleftrightarrow \exists x_1, \ldots, x_m \in \mathbb{N} \;\; W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0 \qquad \texttt{(R)}$$

for some polynomial W with integer coefficients, see [2]. The representation (R) is said to be single-fold, if for any $a_1, \ldots, a_n \in \mathbb{N}$ the equation $W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0$ has at most one solution $(x_1, \ldots, x_m) \in \mathbb{N}^m$. The representation (R) is said to be finite-fold, if for any $a_1, \ldots, a_n \in \mathbb{N}$ the equation $W(a_1, \ldots, a_n, x_1, \ldots, x_m) = 0$ has only finitely many solutions $(x_1, \ldots, x_m) \in \mathbb{N}^m$.

Conjecture 1. *([1, pp. 341–342], [3, p. 42], [4, p. 745]). Every listable set $\mathcal{M} \subseteq \mathbb{N}^n$ $(n \in \mathbb{N} \setminus \{0\})$ has a single-fold Diophantine representation.*

Conjecture 2. *([1, pp. 341–342], [3, p. 42], [4, p. 745]). Every listable set $\mathcal{M} \subseteq \mathbb{N}^n$ $(n \in \mathbb{N} \setminus \{0\})$ has a finite-fold Diophantine representation.*

Let Φ denote the following statement: *the function $\mathbb{N} \ni n \to 2^n \in \mathbb{N}$ eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation*. For $n \in \mathbb{N}$, let

$$g(n) = \begin{cases} 2^n, & \text{if } \Phi \text{ holds} \\ \gamma(n), & \text{otherwise} \end{cases}$$

The function $g : \mathbb{N} \to \mathbb{N}$ is computable if and only if Φ holds. Currently,

$$(\neg\mathcal{K}(\Phi)) \wedge (\neg\mathcal{K}(\neg\Phi)) \wedge (\neg\mathcal{K}(g \; is \; computable)) \wedge (\neg\mathcal{K}(g \; is \; uncomputable))$$

Let Ψ denote the following statement: *the function $\mathbb{N} \ni n \to 2^n \in \mathbb{N}$ eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a finite-fold Diophantine representation*. For $n \in \mathbb{N}$, let

$$h(n) = \begin{cases} 2^n, & \text{if } \Psi \text{ holds} \\ \gamma(n), & \text{otherwise} \end{cases}$$

The function $h : \mathbb{N} \to \mathbb{N}$ is computable if and only if Ψ holds. Currently,

$$(\neg\mathcal{K}(\Psi)) \wedge (\neg\mathcal{K}(\neg\Psi)) \wedge (\neg\mathcal{K}(h \; is \; computable)) \wedge (\neg\mathcal{K}(h \; is \; uncomputable))$$

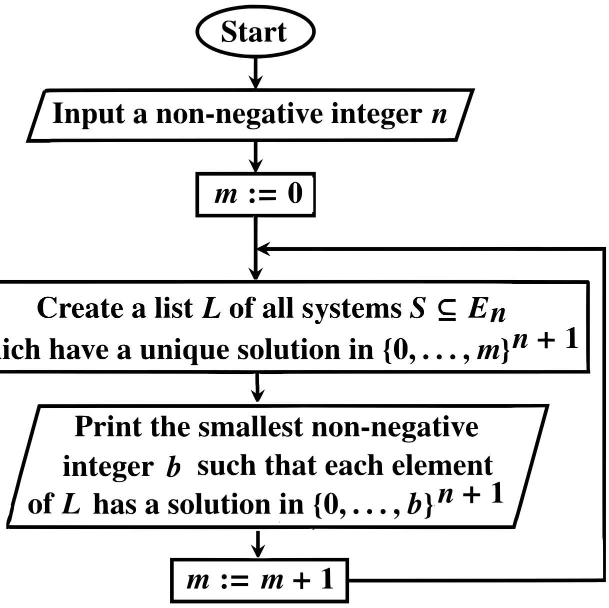
Lemma 3. *The function g is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation. The function h is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a finite-fold Diophantine representation.*

Proof. It follows from Theorem 1. □

For $n \in \mathbb{N}$, $\beta(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a unique solution in $\mathbb{N}^{n+1}$, then this solution belongs to $\{0, \ldots, b\}^{n+1}$.

Theorem 3. *The function $\beta : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. This is proved in [6]. The term *"dominated"* in the title of [6] means *"eventually dominated"*. Flowchart 3 shows a semi-algorithm which computes $\beta(n)$ in the limit, see [6].

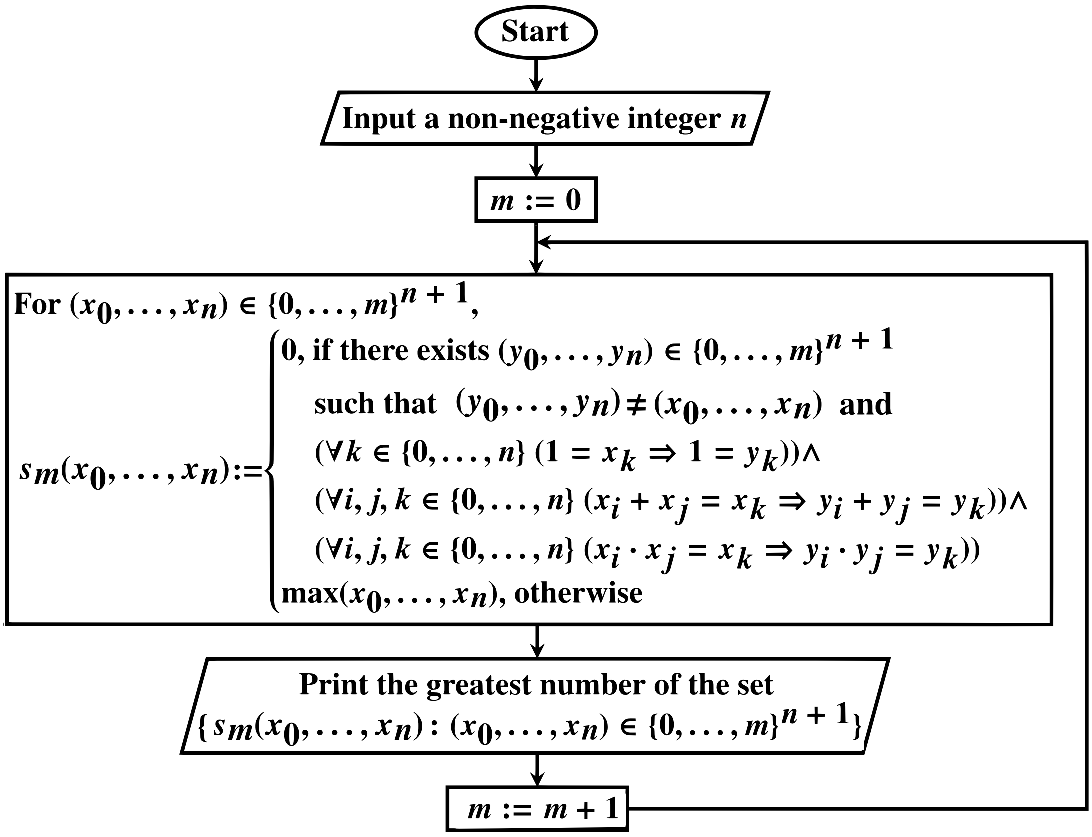


Flowchart 3

A semi-algorithm which computes $\beta(n)$ in the limit

□

Flowchart 4 shows a simpler semi-algorithm which computes $\beta(n)$ in the limit.



Flowchart 4

A simpler semi-algorithm which computes $\beta(n)$ in the limit

Lemma 4. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 4 does not exceed the number printed by Flowchart 3.*

Proof. For every $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k : \ (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup$$
$$\{x_i + x_j = x_k : \ (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$
$$\{x_i \cdot x_j = x_k : \ (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Lemma 5. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 3 does not exceed the number printed by Flowchart 4.*

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $\mathcal{S} \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ is a unique solution of $\mathcal{S}$ in $\{0, \dots, m\}^{n+1}$, then $(a_0, \dots, a_n)$ solves the system of equations

$$\widehat{\mathcal{S}} := \{1 = x_k : \ (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup$$
$$\{x_i + x_j = x_k : \ (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$
$$\{x_i \cdot x_j = x_k : \ (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

By this and the inclusion $\widehat{\mathcal{S}} \supseteq \mathcal{S}$, $\widehat{\mathcal{S}}$ has exactly one solution in $\{0, \dots, m\}^{n+1}$, namely $(a_0, \dots, a_n)$. □

Theorem 4. *For every $n, m \in \mathbb{N}$, Flowcharts 3 and 4 print the same number.*

Proof. It follows from Lemmas 4 and 5. □

Statement 1. *There exists a limit-computable function $f : \mathbb{N} \to \mathbb{N}$ of unknown computability which eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation.*

Proof. Statement 1 follows constructively from Theorem 3 by taking $f = \beta$ because the following conjunction

$$(\neg\mathcal{K}(\beta \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(\beta \text{ is uncomputable}))$$

holds. Statement 1 follows non-constructively from Lemma 3 by taking $f = g$ because the following conjunction

$$(\neg\mathcal{K}(g \text{ is computable})) \ \wedge \ (\neg\mathcal{K}(g \text{ is uncomputable}))$$

holds. □

Since the function γ in Theorem 1 is not computable, Statement 1 does not follow from Theorem 1.

Proposition 2. *Statement 1 strengthens a mathematical theorem. Statement 1 refers to the current mathematical knowledge and may be false in the future. Statement 1 does not express what is currently unproved in mathematics.*

Proof. Statement 1 strengthens Statement 1 without the epistemic condition. The weakened Statement 1 is a theorem which follows from Theorem 1. Statement 1 claims that some mathematically defined function $f : \mathbb{N} \to \mathbb{N}$ satisfies

$$(f \text{ is computable in the limit}) \wedge (\neg\mathcal{K}(f \text{ is computable})) \wedge (\neg\mathcal{K}(f \text{ is uncomputable})) \wedge$$
$$(f \text{ eventually dominates every function } \delta : \mathbb{N} \to \mathbb{N} \text{ with a single-fold Diophantine representation})$$

Conjecture 1 disproves Statement 1. □

Statement 2. *Statement 1 holds for finite-fold Diophantine representations.*

Proof. It follows from Lemma 3 by taking $f = h$ because the following conjunction

$$(\neg\mathcal{K}(h \text{ is computable})) \;\wedge\; (\neg\mathcal{K}(h \text{ is uncomputable}))$$

holds. □

Statement 2 strengthens Statement 1. For Statement 2, there is no known computer program that computes f in the limit.

4 The statement Γ from the title of the article

Statement 3. *In Statement 1, we can require that there exists a computer program which takes as input a non-negative integer n and prints the sequence $\{f_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $f(n)$.*

Proof. Any computer program that implements the semi-algorithm shown in Flowchart 3 or 4 is correct. □

Let Γ denote the following statement: $\exists f : \mathbb{N} \to \mathbb{N}$ *of unknown computability such that f eventually dominates every function $\delta : \mathbb{N} \to \mathbb{N}$ with a single-fold Diophantine representation and there is a computer program which for $n \in \mathbb{N}$ prints the sequence $\{f_i(n)\}_{i=0}^{\infty}$ of non-negative integers converging to $f(n)$.*

Proposition 3. *The statement Γ has all properties from the title of the article.*

Proposition 4. *The statement Γ may hold when $\mathcal{K}(\beta \text{ is computable})$.*

Proof. For $n \in \mathbb{N}$, $\xi(n)$ denotes the smallest $b \in \mathbb{N} \setminus \{0\}$ such that if a system of equations $\mathcal{S} \subseteq E_n$ has a solution in $\mathbb{Q}^{n+1}$, then $\mathcal{S}$ has a solution which consists of rationals whose numerators and denominators belong to $\{-b, \dots, b\}$. The function $\xi : \mathbb{N} \to \mathbb{N}$ is computable in the limit. We skip the proof, which is similar to the proof that γ is computable in the limit. Currently,

$$(\neg\mathcal{K}(\xi \text{ is computable})) \wedge (\neg\mathcal{K}(\xi \text{ is uncomputable}))$$

At any future time, the conjunction

$$(\neg\mathcal{K}(\xi \text{ is computable})) \wedge (\neg\mathcal{K}(\xi \text{ is uncomputable})) \wedge \mathcal{K}(\beta \text{ is computable})$$

may hold and implies the statement Γ with $f = \xi + \beta$. □

Summarizing, the predicate $\mathcal{K}$ can strengthen existential mathematical statements when $\mathcal{K}$ is inserted after $\exists$ and refers to a part of the statement.

5 Predicate $\mathcal{K}$ of the written down mathematical knowledge

In this section, $\mathcal{K}$ denotes both the predicate satisfied by every written down theorem and the finite set of all written down theorems. It changes what is taken as known in mathematics.

Proposition 5. *There exists $k \in \mathbb{N}$ such that the computability of the function*

$$\mathbb{N} \ni n \to k + \gamma(n) \in \mathbb{N}$$

is unknown. For this k, Statements 1 and 2 hold when $f(n) = k + \gamma(n)$.

Proof. It follows from $\operatorname{card}(\mathcal{K}) < \omega$. □

Proposition 5 contradicts Proposition 1 with the right definition of $\mathcal{K}$.

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