A non-recursively enumerable subset of $\mathbb{N}$ which has a short definition in terms of arithmetic

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Abstract

Let $F(x,n)$ denote the formula

$$\exists ab\ \forall i \leqslant n\ \exists swpq\ \forall jv\ \exists eg\ \{(s+w)^2+3w+s=2i\ \wedge\ \langle[j=w\ \vee\ v=q]$$

$$\vee\ [j=3i\ \wedge\ v=p+q]\ \vee\ [j=s\ \wedge\ (v=p\ \vee\ (i=n\ \wedge\ v=q+x))]$$

$$\vee\ [j=3i+1\ \wedge\ v=pq]\ \Rightarrow\ a=v+e+ejb\ \wedge\ v+g=jb\rangle\}$$

from J. P. Jones' article in vol. 43 of J. Symbolic Logic. From the results of Jones' article, it follows that the set $\{n \in \mathbb{N} : \neg F(n,n)\}$ is co-recursively enumerable and not recursively enumerable. We prove that the set

$$W=\{n\in\mathbb{N}:\ \exists p,q\in\mathbb{N}\ ((2n=(p+q)(p+q+1)+2q)\ \wedge$$

$$\forall(x_0,\ldots,x_p)\in\mathbb{N}^{p+1}\ \ \exists(y_0,\ldots,y_p)\in\{0,\ldots,q\}^{p+1}$$

$$((\forall j,k\in\{0,\ldots,p\}\ (x_j+1=x_k\Rightarrow y_j+1=y_k))\ \wedge$$

$$(\forall i,j,k\in\{0,\ldots,p\}\ (x_i\cdot x_j=x_k\Rightarrow y_i\cdot y_j=y_k)))))\}$$

is co-recursively enumerable and not recursively enumerable. Let $\beta:\mathbb{N}^3\to\mathbb{N}$ denote Gödel's β function. For $x_1,x_2,x_3\in\mathbb{N}$, $\beta(x_1,x_2,x_3)$ equals the remainder after integer division of x_1 by $1+(x_3+1)\cdot x_2$. We prove that the set W consists of all $n\in\mathbb{N}$ such that

$$\forall u,v\in\mathbb{N}\ \exists a,b,p,q\in\mathbb{N}\ ((2n=(p+q)(p+q+1)+2q)\ \wedge\ \forall i,j,k\in\{0,\ldots,p\}$$

$$((\beta(a,b,i)\leqslant q)\ \wedge\ (\beta(u,v,j)+1=\beta(u,v,k)\Rightarrow\beta(a,b,j)+1=\beta(a,b,k))\ \wedge$$

$$(\beta(u,v,i)\cdot\beta(u,v,j)=\beta(u,v,k)\Rightarrow\beta(a,b,i)\cdot\beta(a,b,j)=\beta(a,b,k))))$$

We express the above formula in Peano arithmetic.

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Let $F(x,n)$ denote the formula

$$\exists ab\ \forall i \leqslant n\ \exists swpq\ \forall jv\ \exists eg\ \{(s+w)^2+3w+s=2i\ \wedge\ \langle[j=w\ \vee\ v=q]$$

$$\vee\ [j=3i\ \wedge\ v=p+q]\ \vee\ [j=s\ \wedge\ (v=p\ \vee\ (i=n\ \wedge\ v=q+x))]$$

$$\vee\ [j=3i+1\ \wedge\ v=pq]\ \Rightarrow\ a=v+e+ejb\ \wedge\ v+g=jb\rangle\}$$

from [2, p. 336]. From the results of [2], it follows that the set $\{n \in \mathbb{N} : \neg F(n,n)\}$ is co-recursively enumerable and not recursively enumerable. In this article, we define another non-recursively enumerable subsets of $\mathbb{N}$ which have a short definition in terms of arithmetic.

Semi-algorithms differ from algorithms, as they may not terminate.

Definition 1. *(cf. [4, pp. 233–235]). A computation in the limit of a function* $f:\mathbb{N}\to\mathbb{N}$ *is a semi-algorithm which takes as input a non-negative integer* n *and for every* $m\in\mathbb{N}$ *prints a non-negative integer* $\xi(n,m)$ *such that* $\lim\limits_{m\to\infty}\xi(n,m)=f(n)$.

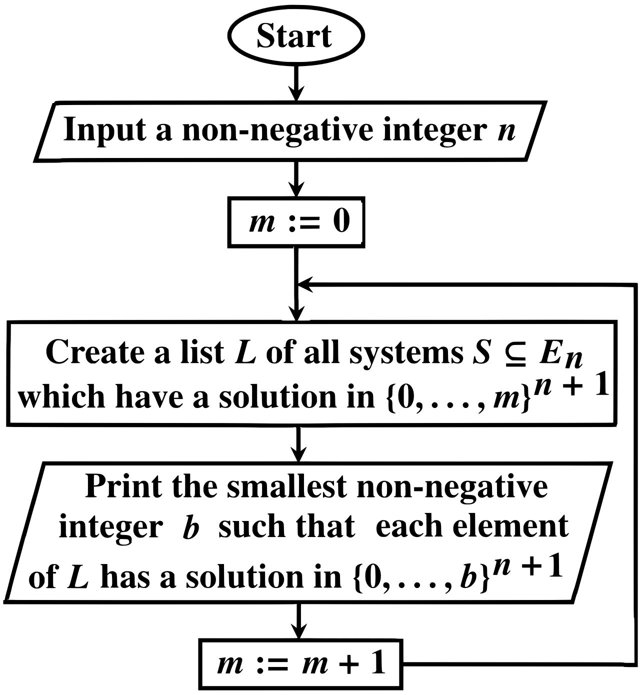
By Definition 1, a function $f:\mathbb{N}\to\mathbb{N}$ is computable in the limit when there exists an infinite computation which takes as input a non-negative integer n and prints a non-negative integer on each iteration and prints $f(n)$ on each sufficiently high iteration.

For $n\in\mathbb{N}$, let

$$E_n=\{1=x_k,\ x_i+x_j=x_k,\ x_i\cdot x_j=x_k:\ i,j,k\in\{0,\dots,n\}\}$$

For $n\in\mathbb{N}$, $f(n)$ denotes the smallest $b\in\mathbb{N}$ such that if a system of equations $S\subseteq E_n$ has a solution in $\mathbb{N}^{n+1}$, then S has a solution in $\{0,\dots,b\}^{n+1}$. The function $f:\mathbb{N}\to\mathbb{N}$ is computable in the limit and eventually dominates every computable function $g:\mathbb{N}\to\mathbb{N}$, see [6]. The term *"dominated"* in the title of [6] means *"eventually dominated"*.

Theorem 1. *([6]). Flowchart 1 shows a semi-algorithm which computes* $f(n)$ *in the limit.*



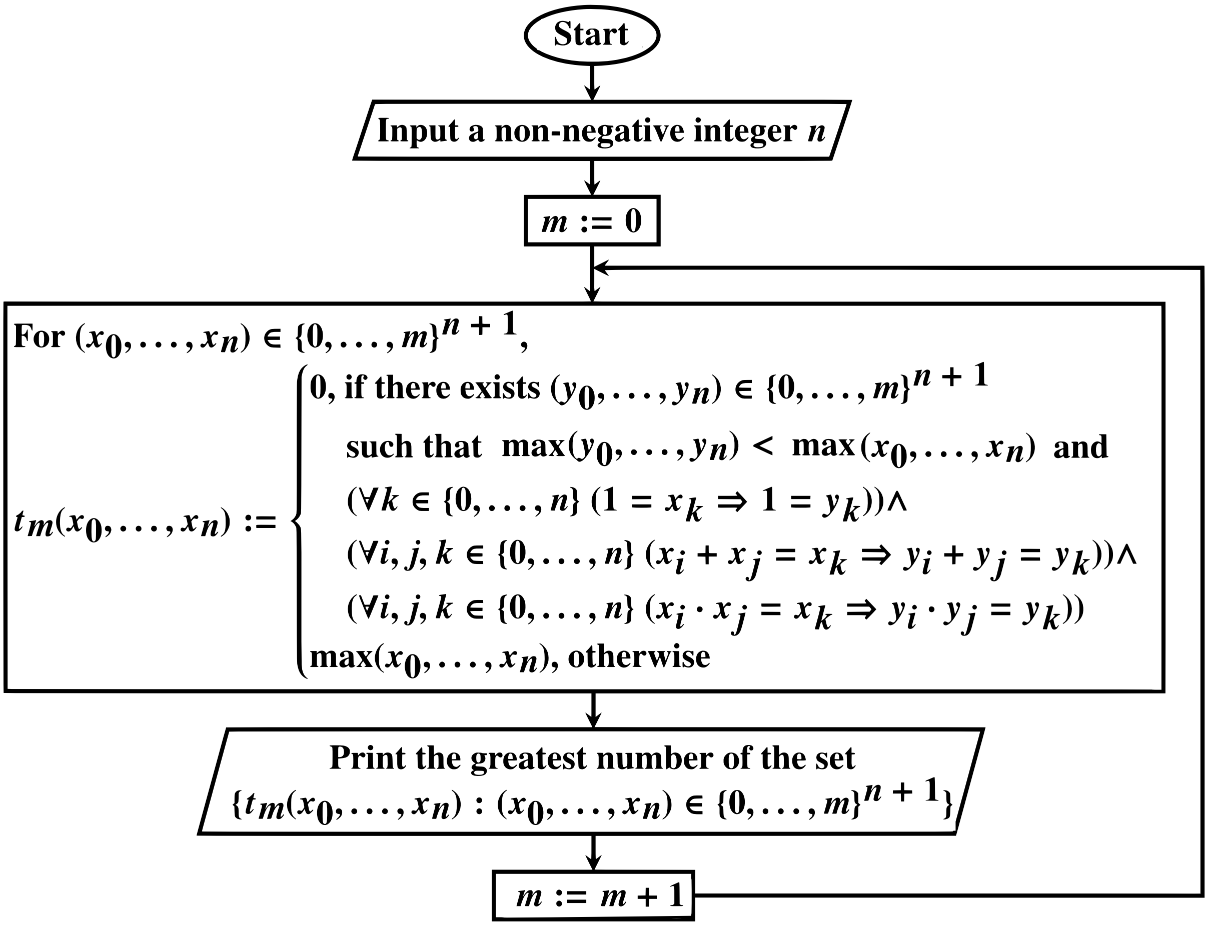
Flowchart 1

A semi-algorithm which computes $f(n)$ in the limit

Definition 2. *An approximation of a tuple* $(x_0,\dots,x_n)\in\mathbb{N}^{n+1}$ *is a tuple* $(y_0,\dots,y_n)\in\mathbb{N}^{n+1}$ *such that*

$$(\forall k\in\{0,\dots,n\}\ (1=x_k\Rightarrow 1=y_k))\ \wedge$$
$$(\forall i,j,k\in\{0,\dots,n\}\ (x_i+x_j=x_k\Rightarrow y_i+y_j=y_k))\ \wedge$$
$$(\forall i,j,k\in\{0,\dots,n\}\ (x_i\cdot x_j=x_k\Rightarrow y_i\cdot y_j=y_k))$$

Flowchart 2 shows a simpler semi-algorithm which computes $f(n)$ in the limit.



Flowchart 2

A simpler semi-algorithm which computes $f(n)$ in the limit

Lemma 1. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 2 does not exceed the number printed by Flowchart 1.*

Proof. For every $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$,

$$E_n \supseteq \{1 = x_k :\ (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Lemma 2. *For every $n, m \in \mathbb{N}$, the number printed by Flowchart 1 does not exceed the number printed by Flowchart 2.*

Proof. Let $n, m \in \mathbb{N}$. For every system of equations $S \subseteq E_n$, if $(a_0, \dots, a_n) \in \{0, \dots, m\}^{n+1}$ and $(a_0, \dots, a_n)$ solves S, then $(a_0, \dots, a_n)$ solves the following system of equations:

$$\{1 = x_k :\ (k \in \{0, \dots, n\}) \wedge (1 = a_k)\} \cup$$

$$\{x_i + x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i + a_j = a_k)\} \cup$$

$$\{x_i \cdot x_j = x_k :\ (i, j, k \in \{0, \dots, n\}) \wedge (a_i \cdot a_j = a_k)\}$$

□

Theorem 2. *For every* $n, m \in \mathbb{N}$*, Flowcharts 1 and 2 print the same number.*

Proof. It follows from Lemmas 1 and 2. $\square$

Corollary 1. *For every* $n, m \in \mathbb{N}$*, Flowcharts 1 and 2 print the smallest* $b \in \{0, \ldots, m\}$ *such that every tuple* $(x_0, \ldots, x_n) \in \{0, \ldots, m\}^{n+1}$ *possesses an approximation in* $\{0, \ldots, b\}^{n+1}$*.*

Theorem 3. *For every* $n \in \mathbb{N}$*,* $f(n)$ *is the smallest* $b \in \mathbb{N}$ *such that every tuple* $(x_0, \ldots, x_n) \in \mathbb{N}^{n+1}$ *possesses an approximation in* $\{0, \ldots, b\}^{n+1}$*.*

Proof. It follows from Theorem 1 and Corollary 1. $\square$

Theorem 4. *No algorithm takes as input non-negative integers* n *and* m *and decides whether or not*

$$\forall (x_0, \ldots, x_n) \in \mathbb{N}^{n+1} \;\; \exists (y_0, \ldots, y_n) \in \{0, \ldots, m\}^{n+1}$$
$$((\forall k \in \{0, \ldots, n\} \; (1 = x_k \Rightarrow 1 = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, n\} \; (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, n\} \; (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)))$$

Proof. Since the function f is not computable, it follows from Theorem 3. $\square$

Lemma 3. *([3]). The function*

$$\mathbb{N}^2 \ni (p, q) \to \frac{1}{2}(p+q)(p+q+1) + q \in \mathbb{N}$$

is bijective.

Theorem 5. *No algorithm takes as input a non-negative integer* n *and decides whether or not*

$$\exists p, q \in \mathbb{N} \; ((2n = (p+q)(p+q+1) + 2q)) \wedge$$
$$\forall (x_0, \ldots, x_p) \in \mathbb{N}^{p+1} \;\; \exists (y_0, \ldots, y_p) \in \{0, \ldots, q\}^{p+1}$$
$$((\forall k \in \{0, \ldots, p\} \; (1 = x_k \Rightarrow 1 = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, p\} \; (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, p\} \; (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))))$$

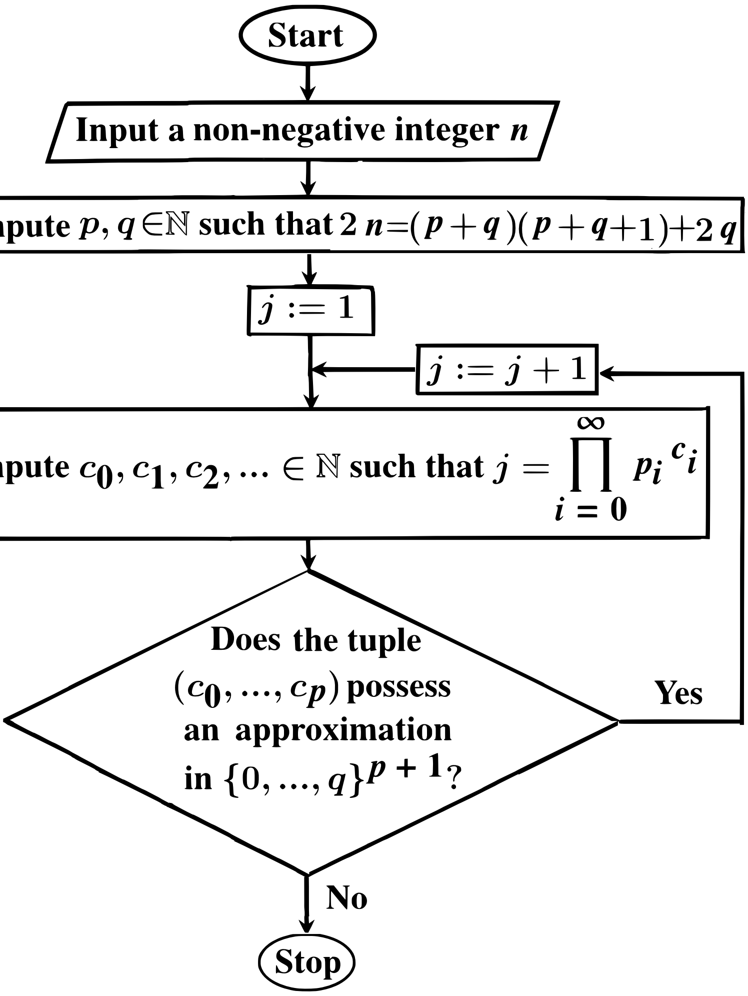
Proof. It follows from Theorem 4 and Lemma 3. $\square$

Let

$$T = \{n \in \mathbb{N} : \; \exists p, q \in \mathbb{N} \; ((2n = (p+q)(p+q+1) + 2q) \wedge$$
$$\forall (x_0, \ldots, x_p) \in \mathbb{N}^{p+1} \;\; \exists (y_0, \ldots, y_p) \in \{0, \ldots, q\}^{p+1}$$
$$((\forall k \in \{0, \ldots, p\} \; (1 = x_k \Rightarrow 1 = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, p\} \; (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k)) \wedge$$
$$(\forall i, j, k \in \{0, \ldots, p\} \; (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)))))\}$$

Theorem 6. *The set* $\mathbb{N} \setminus T$ *is recursively enumerable.*

Proof. For $i \in \mathbb{N}$, let p_i denote the i-th prime number. Flowchart 3 shows a semi-algorithm which takes as input $n \in \mathbb{N}$ and terminates if and only if $n \in \mathbb{N} \setminus T$.



Flowchart 3

A semi-algorithm which takes as input $n \in \mathbb{N}$ and terminates if and only if $n \in \mathbb{N} \setminus T$

□

Theorem 7. *The set* T *is not recursively enumerable.*

Proof. It follows from Theorems 5 and 6. □

Lemma 4. *([5, p. 110]). For non-negative integers, the equation* $x + y = z$ *is equivalent to a system which consists of equations of the forms* $v + 1 = w$ *and* $u \cdot v = w$.

For $n \in \mathbb{N}$, $h(n)$ denotes the smallest $b \in \mathbb{N}$ such that if a system of equations $S \subseteq \{x_j + 1 = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{0, \ldots, n\}\}$ has a solution in $\mathbb{N}^{n+1}$, then S has a solution in $\{0, \ldots, b\}^{n+1}$. From Lemma 4 and [6], it follows that the function $h : \mathbb{N} \to \mathbb{N}$ is computable in the limit and eventually dominates every computable function $g : \mathbb{N} \to \mathbb{N}$.

Theorem 8. *No algorithm takes as input non-negative integers n and m and decides whether or not*

$$\forall (x_0,\ldots,x_n) \in \mathbb{N}^{n+1}\ \exists (y_0,\ldots,y_n) \in \{0,\ldots,m\}^{n+1}$$
$$((\forall j,k \in \{0,\ldots,n\}\ (x_j+1=x_k \Rightarrow y_j+1=y_k)) \wedge$$
$$(\forall i,j,k \in \{0,\ldots,n\}\ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)))$$

Proof. It holds because the function h is not computable, and for every $n \in \mathbb{N}$, $h(n)$ is the smallest $b \in \mathbb{N}$ such that

$$\forall (x_0,\ldots,x_n) \in \mathbb{N}^{n+1}\ \exists (y_0,\ldots,y_n) \in \{0,\ldots,b\}^{n+1}$$
$$((\forall j,k \in \{0,\ldots,n\}\ (x_j+1=x_k \Rightarrow y_j+1=y_k)) \wedge$$
$$(\forall i,j,k \in \{0,\ldots,n\}\ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k)))$$

□

Theorem 9. *No algorithm takes as input a non-negative integer n and decides whether or not*

$$\exists p,q \in \mathbb{N}\ ((2n=(p+q)(p+q+1)+2q) \wedge$$
$$\forall (x_0,\ldots,x_p) \in \mathbb{N}^{p+1}\ \exists (y_0,\ldots,y_p) \in \{0,\ldots,q\}^{p+1}$$
$$((\forall j,k \in \{0,\ldots,p\}\ (x_j+1=x_k \Rightarrow y_j+1=y_k)) \wedge$$
$$(\forall i,j,k \in \{0,\ldots,p\}\ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k))))$$

Proof. It follows from Theorem 8 and Lemma 3. □

Let W denote the algorithmically undecidable subset of $\mathbb{N}$ considered in Theorem 9. Similarly as in Theorem 6, the set $\mathbb{N} \setminus W$ is recursively enumerable. Similarly as in Theorem 7, the set W is not recursively enumerable. Let $\beta : \mathbb{N}^3 \to \mathbb{N}$ denote Gödel's β function, see [1]. For $x_1,x_2,x_3 \in \mathbb{N}$, $\beta(x_1,x_2,x_3)$ equals the remainder after integer division of x_1 by $1+(x_3+1)\cdot x_2$.

Lemma 5. *([1]). If $(d_0,\ldots,d_p) \in \mathbb{N}^{p+1}$, then $\exists z_1,z_2 \in \mathbb{N}\ \forall l \in \{0,\ldots,p\}\ \beta(z_1,z_2,l)=d_l$.*

By Lemma 5, the set W consists of all $n \in \mathbb{N}$ such that

$$\forall u,v \in \mathbb{N}\ \exists a,b,p,q \in \mathbb{N}\ ((2n=(p+q)(p+q+1)+2q)\ \wedge\ \forall i,j,k \in \{0,\ldots,p\}$$
$$((\beta(a,b,i) \leqslant q)\ \wedge\ (\beta(u,v,j)+1=\beta(u,v,k) \Rightarrow \beta(a,b,j)+1=\beta(a,b,k)) \wedge$$
$$(\beta(u,v,i)\cdot\beta(u,v,j)=\beta(u,v,k) \Rightarrow \beta(a,b,i)\cdot\beta(a,b,j)=\beta(a,b,k))))$$

In Peano arithmetic, the above formula expresses that

$$\forall\ u\ v\ \exists\ a\ b\ p\ q\ ((2n=(p+q)(p+q+1)+2q)\ \wedge$$
$$\forall\ i\ j\ k\ (((i \leqslant p) \wedge (j \leqslant p) \wedge (k \leqslant p))\ \Rightarrow\ \exists\ x_1\ x_2\ x_3\ \widetilde{x_1}\ \widetilde{x_2}\ \widetilde{x_3}\ y_1\ y_2\ y_3\ \widetilde{y_1}\ \widetilde{y_2}\ \widetilde{y_3}$$
$$((x_1 < 1+(i+1)\cdot v) \wedge (x_2 < 1+(j+1)\cdot v) \wedge (x_3 < 1+(k+1)\cdot v) \wedge$$
$$(y_1 < 1+(i+1)\cdot b) \wedge (y_2 < 1+(j+1)\cdot b) \wedge (y_3 < 1+(k+1)\cdot b) \wedge$$
$$(u=\widetilde{x_1}\cdot(1+(i+1)\cdot v)+x_1) \wedge (u=\widetilde{x_2}\cdot(1+(j+1)\cdot v)+x_2) \wedge (u=\widetilde{x_3}\cdot(1+(k+1)\cdot v)+x_3) \wedge$$
$$(a=\widetilde{y_1}\cdot(1+(i+1)\cdot b)+y_1) \wedge (a=\widetilde{y_2}\cdot(1+(j+1)\cdot b)+y_2) \wedge (a=\widetilde{y_3}\cdot(1+(k+1)\cdot b)+y_3) \wedge$$
$$(y_1 \leqslant q) \wedge (x_2+1=x_3 \Rightarrow y_2+1=y_3) \wedge (x_1\cdot x_2=x_3 \Rightarrow y_1\cdot y_2=y_3))))$$

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